UDC 531.36

ON STABILITY IN THE PRESENCE OF SEVERAL RESONANCES

PMM Vol. 41, №3, 1977, pp. 422-429
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(Received June 22, 1976)

Liapunov stability of the zero solution of an autonomous system of ordinary differential equations is investigated in the case when the characteristic equations of the related linearized system has only different pure imaginary roots that satisfy some linear integral relations or, putting it differently, the condition of inner resonance. The simultaneous presence of several resonance relationships, which was touched upon earlier in [1, 3], is considered here. The normal form of a system that contains the first nonlinear terms for an arbitrary number of noninteracting, as well as interacting resonances of an odd order is formulated on the basis of results obtained in [3]. The case of resonance interaction in which the necessary and sufficient conditions of the model system stability are reduced to conditions formulated in [3] for a single resonance. Necessary conditions of stability are defined for the most general case of resonance interaction. As an example of the complex mechanical system in which resonance interaction may occur, the translational-rotational motion of a geo-stationary satellite vehicle, which can hover for a fairly extended time over any point of the Earth, is considered. All resonance modes, including some of the considered cases of resonance interaction in the region where the necessary stability conditions are satisfied, were determined with the use of a computer.

Let us consider the problem of stability of the zero solution of the system of equations

$$x_*^{\cdot} = Ax_* + X_* (x_*), \quad x_*^{\cdot} = dx_* / dt \tag{0.1}$$

$$x_* = (x_1^*, \ldots, x_{2q}^*), \quad X_* = (X_1^*, \ldots, X_{2q}^*), \quad X_* (0) = 0$$

where x_* and X_* are $2q$-dimensional vectors of the Euclidean space E_{2q}, A is a constant square matrix with only pure imaginary eigenvalues $\pm \lambda_s$ $(\lambda_s^2 < 0$ $(s = 1, 2, \ldots, q)$, among which there are no multiples and $X_* (x_*)$ are holomorphic functions whose expansions in powers of x_* begin with m-th order forms.

Let system (0.1) have an intrinsic resonance of order k i.e. the relationship of the form $\langle \Lambda, P \rangle = 0, \quad P = (p_1, \ldots, p_q), \quad p_s \geqslant 0$

$$\Lambda = (\lambda_1, \ldots, \lambda_q), \quad | P | = p_1 + \ldots + p_q = k, \quad k = m + 1$$

where V is the vector of eigenvalues of matrix A and p_s are mutually disjoint integers, are satisfied.

Systems with only one resonance relationship of the indicated form were investigated in [3-5]. Let us consider more complex systems in which several resonance relationships can exist simultaneously. Certain particular cases of that problem were analyzed in [1, 2].

The object of the present work is the extension of the results obtained in [1, 2] to any kind and number of resonances for odd k.

Let us consider the following form of the input system (0.1):

$$x^{\cdot} = \lambda x + \sum_{l=m\geqslant 2}^{\infty} X^{(l)}(x, y)$$
$$y^{\cdot} = -\lambda y + \sum_{l=m\geqslant 2}^{\infty} Y^{(l)}(x, y) \tag{0.2}$$
$$x = (x_1, \ldots, x_q) \quad y = (y_1, \ldots, y_q) \quad \lambda = \{\lambda_1, \ldots, \lambda_q\}$$

which is obtained by the complex linear transformation described in [6], and in which x and y are complex conjugate vectors, λ is a diagonal matrix, and $X^{(l)}$ and $Y^{(l)}$ are complex conjugate vector functions whose components $X_s^{(l)}$, $Y_s^{(l)}$ ($s = 1, 2, \ldots, q$) are l-th order forms of x and y

1. **The case of independent resonances.** Let system (0.2.) have μ resonance relationships of the form

$$\langle \Lambda_\nu, P_\nu \rangle = 0, \quad \nu = 1, 2, \ldots, \mu$$
$$\Lambda_\nu = (\lambda_{\nu 1}, \lambda_{\nu 2}, \ldots, \lambda_{\nu n_\nu}), \quad |P_1| = \ldots = |P_\mu| = k \tag{1.1}$$
$$n_1 + n_2 + \ldots + n_\mu = n \leqslant q$$

where Λ_ν is the ν-th vector component of vector $\Lambda = (\Lambda_1, \ldots, \Lambda_\mu)$, and P_ν is an integral vector with positive components $p_{\nu 1}, \ldots, p_{\nu n_\nu}$.

By applying to system (0.2) the nonlinear normalizing transformation described in [3] we obtain

$$r_{\nu s}^{\cdot} = 2Q_{\nu s}(\theta_\nu) R_\nu + \ldots \tag{1.2}$$
$$\theta_\nu^{\cdot} = R_\nu \sum_{j=1}^{n_\nu} \frac{p_{\nu s}}{r_{\nu s}} Q'_{\nu s}(\theta_\nu) + \ldots$$
$$s = 1, 2, \ldots, n_\nu; \quad \nu = 1, 2, \ldots, \mu$$
$$r_\alpha^{\cdot} = O(r^{1/2(k+1)}), \quad \theta_\alpha^{\cdot} = O(r^{1/2(k+1)}), \quad \alpha = n+1, \ldots q$$
$$\theta_\nu = p_{\nu 1}\theta_{\nu 1} + \ldots + p_{\nu n_\nu}\theta_{\nu n_\nu}, \quad R_\nu = \sqrt{\prod_{j=1}^{n_\nu} r_{\nu j}^{p_{\nu j}}}$$
$$Q_{\nu s} = a_{\nu s}\cos\theta_\nu + b_{\nu s}\sin\theta_\nu, \quad Q'_{\nu s}(\theta_\nu) = \frac{dQ_{\nu s}}{d\theta_\nu}$$
$$r = \sum_{\nu=1}^{\mu} (r_{\nu 1} + \ldots + r_{\nu n_\nu}) + r_{n+1} + \ldots + r_q$$

where $r_{\nu s}$, $\theta_{\nu s}$ r_α, θ_α are polar coordinates and the implicit terms are of order not lesser than the $k+1$-st with respect to r^2.

Thus the model system (derived from (1.2) by the rejection of implicit terms) decomposes into a nonresonance system (in variables r_α and θ_ν) and μ independent resonant subsystems (in variables $r_{\nu s}$ and θ_ν), which were the subject of detailed analysis in [3].

Theorem 1. If the trivial solution of the model system (1.2) is to be stable it is necessary and sufficient that the trivial solutions of each resonance subsystem are stable.

The necessary and sufficient conditions for the stability of resonance subsystems are given in [3] in the form of inequalities that are to be satisfied by the normal form of coefficients $a_{\nu s}$ and $b_{\nu s}$.

2. **The case of resonance interaction.** Let us consider the case in which resonance relationships (1.1) have common frequencies. We begin with the simplest case in which each resonance relationship has only one common frequency λ_0, and, consequently, the condition of intrinsic resonance can be written as

$$\lambda_0 p_{\nu 0} + \langle \Lambda_\nu, P_\nu \rangle = 0, \quad \nu = 1, 2, \ldots, \mu \tag{2.1}$$
$$p_{\nu 0} + | P_\nu | = k, \quad k = m + 1$$

where Λ_ν and P_ν retain their original meaning. As previously, we assume that from the overall number q of frequencies only $n = 1 + n_1 + n_2 + \ldots + n_\mu \leqslant q$ frequencies participate in the resonance.

Applying successively to the system the set of transformations shown in [3] and taking into account (2.1), we can obtain for system (0.2) in polar coordinates $r_0, \theta_0; r_{\nu s}, \theta_{\nu s}$ to within the first nonlinear terms the following normal form:

$$\begin{gathered}
r_0^{\cdot} = 2 \sum_{\nu=1}^{\mu} R_\nu Q_{\nu 0}(\theta_\nu) + \ldots, \quad r_{\nu s} = 2 R_\nu Q_{\nu s}(\theta_\nu) + \ldots \\
\theta_\nu^{\cdot} = R_\nu \sum_{s=1}^{n_\nu} \frac{p_{\nu s}}{r_{\nu s}} Q_{\nu s}'(\theta_\nu) + \sum_{\beta=1}^{\mu} \frac{p_{\nu 0}}{r_0} R_\beta Q_{\beta 0}'(\theta_\beta) + \ldots \\
r_\alpha^{\cdot} = O(r^{1/2(k+1)}), \quad r_\alpha \theta_\alpha^{\cdot} = O(r^{1/2(k+1)}) \\
s = 1, 2, \ldots, n_\nu, \quad \nu = 1, 2, \ldots, \mu; \quad \alpha = n+1, \ldots, q \\
R_\nu^2 = r_0^{p_{\nu 0}} \prod_{j=1}^{n_\nu} r_{\nu j}^{p_{\nu j}}, \quad Q_{\nu s}(\theta_\nu) = a_{\nu s} \cos\theta_\nu + b_{\nu s} \sin\theta_\nu \\
\theta_\nu = p_{\nu 0}\theta_0 + p_{\nu 1}\theta_{\nu 1} + \ldots + p_{\nu n_\nu}\theta_{\nu n_\nu} \\
r = r_0 + \sum_{\nu=1}^{\mu} (r_{\nu 1} + \ldots + r_{\nu n_\nu}) + r_{n+1} + \ldots + r_q
\end{gathered} \tag{2.2}$$

Equations (2.2) show that the problems of stability of the model system trivial solution is again reduced to that of stability of the first $n + 1$ equations which constitute the resonance subsystem. However, unlike independent resonances, this subsystem does not decompose into μ independent subsystems. It is nevertheless possible to obtain the necessary and sufficient conditions for the stability of the model system. We shall show that such conditions require the existence of the constant-sign integral

$$\Phi \equiv c_0 r_0 + \sum_{\nu=1}^{\mu} (c_{\nu 1} r_{\nu 1} + \ldots + c_{\nu n_\nu} r_{\nu n_\nu}) + \sum_{\alpha=n+1}^{q} r_\alpha = \text{const} \tag{2.3}$$
$$c_0, c_{\nu 1}, \ldots, c_{\nu n_\nu} > 0$$

Equating to zero the derivative of (2.3), by virtue of the model system (2.2) we obtain equations

$$c_0 a_{\nu 0} + c_{\nu 1} a_{\nu 1} + \ldots + c_{\nu n_\nu} a_{\nu n_\nu} = 0$$
$$c_0 b_{\nu 0} + c_{\nu 1} b_{\nu 1} + \ldots + c_{\nu n_\nu} b_{\nu n_\nu} = 0, \quad \nu = 1, 2, \ldots \mu \tag{2.4}$$

which must be satisfied by the constants $c_0, c_{\nu 1}, \ldots, c_{\nu n_\nu}$. Thus the existence of a positive solution of system (2.4.) is a sufficient condition of existence of the integral (2.3).

First, let us consider the nondegenerate case in which each of matrices

$$A_\nu = \begin{Vmatrix} a_{\nu 0} & a_{\nu 1} & \ldots & a_{\nu n_\nu} \\ b_{\nu 0} & b_{\nu 1} & \ldots & b_{\nu n_\nu} \end{Vmatrix}, \quad \nu = 1, 2, \ldots, \mu$$

the rank of $A_\nu = 2$ и $n_\nu \geqslant 2$.

Composing all possible matrices

$$A_{\nu_{\alpha\beta\gamma}} = \begin{Vmatrix} a_{\nu\alpha} & a_{\nu\beta} & a_{\nu\gamma} \\ b_{\nu\alpha} & b_{\nu\beta} & b_{\nu\gamma} \end{Vmatrix}, \quad \alpha, \beta, \gamma = 0, 1, 2, \ldots, n_\nu$$

it is possible to show that the necessary and sufficient condition of existence of the indicated above solution is of the same form as in the case of a single resonance [3, 5], namely

$$\operatorname{sign} \begin{vmatrix} a_{\nu\alpha} & a_{\nu\beta} \\ b_{\nu\alpha} & b_{\nu\beta} \end{vmatrix} = \operatorname{sign} \begin{vmatrix} a_{\nu\beta} & a_{\nu\gamma} \\ b_{\nu\beta} & b_{\nu\gamma} \end{vmatrix} = \operatorname{sign} \begin{vmatrix} a_{\nu\gamma} & a_{\nu\alpha} \\ b_{\nu\gamma} & b_{\nu\alpha} \end{vmatrix} \tag{2.5}$$

The fulfilment of the above condition for any single value of ν means that the model system would be stable, if only one resonance relation is satisfied for the indicated value of ν. Such resonance (which preserves the indifferent stability of the model system) will be called weak resonance. If, however, such resonance (in the absence of any other) results in the instability of the models system it will be called strong resonance. If the rank of $A_\nu = 1$ conditions (2.5) lose their meaning. But, then as implied by (2.4), the necessary and sufficient condition of existence of a positive solution is that matrix A_ν contains a pair of elements $a_{\nu\alpha}$, $a_{\nu\beta}$ (or $b_{\nu\alpha}$, $b_{\nu\beta}$); α, and $\beta = 0, 1, 2, \ldots, n_\nu$ of opposite signs. This, in particular, occurs with Hamiltonian systems, as well as in the case of other systems that contain two-frequency resonances, to which in system (2.4) correspond pairs of equations each containing two unknown c_0 and $c_{\nu 1}$.

Let us show that when system (2.4) has no positive solution even for only one value $\nu = \varkappa$ the trivial equation of the model system (2.2) is unstable.

We exclude from the analysis the case of particular interaction when among resonances (2.1) there is a weak one for which condition

$$n_\nu = | P_\nu | = 1 \tag{2.6}$$

is satisfied (in the absence of multiple roots + of the input system only one such resonance is, evidently, possible). This case requires special investigation.

Setting in (2.2) $r_{\nu s} = 0$; $s = 1, 2, \ldots, n_\nu$; $\nu = 1, 2, \ldots, \varkappa - 1, \varkappa +$

$1, \ldots, \mu$ for the resonance subsystem we obtain

$$\begin{aligned}
\dot{r}_0 &= 2R_\varkappa Q_{\varkappa 0}(\theta_\varkappa) \\
r_{\varkappa s}^{\cdot} &= 2R_\varkappa Q_{\varkappa s}(\theta_\varkappa), \quad s = 1, 2, \ldots, n_\varkappa \\
\theta_\varkappa^{\cdot} &= R_\varkappa\left[\frac{p_{\varkappa 0}}{r_0} Q'_{\varkappa 0}(\theta_\varkappa) + \sum_{j=1}^{n_\varkappa} \frac{p_{\varkappa j}}{r_{\varkappa j}} Q'_{\varkappa j}(\theta_\varkappa)\right]
\end{aligned} \tag{2.7}$$

The failure to satisfy conditions (2.5) for $\nu = \varkappa$ implies the instability of the trivial solution of system (2.7) [3, 5]. This can be shown with the use of Chetaev's function or by the direct construction of the unstable particular solution. The same can be done if the rank of $A_\varkappa = 1$ and among the elements of matrix $A_\varkappa$ there is not a single pair $a_{\varkappa\alpha}, a_{\varkappa\beta}$ (or $b_{\varkappa\alpha}, b_{\varkappa\beta}$); α and $\beta = 0, 1, 2, \ldots, n_\varkappa$ of opposite signs.

On the basis of the above we can formulate the following theorem.

Theorem 2. For the stability of the trivial solution of the model system (0.2) with resonance (2.1) it is necessary and sufficient that each of the resonances is weak.

Let us now consider the most general case, when each of the resonance relations contain two or more common frequencies, i.e. we assume that the first n $(n \leqslant q)$ eigenvalues satisfy resonance relations of the form

$$\begin{aligned}
&\langle \Lambda_0, G_\nu\rangle + \langle \Lambda_\nu, P_\nu\rangle = 0, \quad \nu = 1, 2, \ldots, \mu \\
&\Lambda_0 = (\lambda_1, \lambda_2, \ldots, \lambda_{n_0}), \quad \Lambda_\nu = (\lambda_{\nu 1}, \lambda_{\nu 2}, \ldots, \lambda_{\nu n_\nu}) \\
&G_\nu = (g_{\nu 1}, g_{\nu 2}, \ldots, g_{\nu n_0}), \quad P_\nu = (p_{\nu 1}, p_{\nu 2}, \ldots, p_{\nu n_\nu}) \\
&n_0 + n_1 + n_2 + \ldots + n_\mu = n, \quad |G_\nu| + |P_\nu| = k, \quad k = m + 1
\end{aligned} \tag{2.8}$$

where Λ_0 is the vector component of eigenvalues that is common for all resonance relations, Λ_ν is the vector component of eigenvalues appearing only in the ν-th resonant relation, and G_ν and P_ν are vectors of dimensions n_0 and n_ν respectively, with positive integral components.

Using the transformations described in [3] for odd k we obtain in polar coordinates the following normal form of system (0.2):

$$\begin{aligned}
&r_s^{\cdot} = 2\sum_{\nu=1}^{\mu} R_\nu Q_\nu^*(\theta_\nu) + \ldots, \quad s = 1, 2, \ldots, n_0 \\
&r_{\nu\sigma} = 2R_\nu Q_{\nu\sigma}(\theta_\nu) + \ldots, \quad \sigma = 1, 2, \ldots, n_\nu \\
&\theta_\nu^{\cdot} = R_\nu \sum_{\sigma=1}^{n_\nu} \frac{p_{\nu\sigma}}{r_{\nu\sigma}} Q'_{\nu\sigma}(\theta_\nu) + \sum_{\beta=1}^{\mu}\sum_{s=1}^{n_0} \frac{g_{\beta s}}{r_s} R_\beta Q_{\beta s}^{*\prime}(\theta_\beta) + \ldots \\
&\theta_\nu = \theta_1 g_{\nu 1} + \ldots + \theta_{n_0} g_{\nu n_0} + \theta_{\nu 1} p_{\nu 1} + \ldots + \theta_{\nu n_\nu} p_{\nu n_\nu} \\
&R_\nu^2 = \prod_{j=1}^{n_0} r_j^{g_{\nu j}} \prod_{i=1}^{n_\nu} r_{\nu i}^{p_{\nu i}}, \quad Q_{\nu s}^*(\theta_\nu) = a_{\nu s}^* \cos\theta_\nu + b_{\nu s}^* \sin\theta_\nu
\end{aligned} \tag{2.9}$$

$$Q_{\nu\sigma}(\theta_\nu) = a_{\nu\sigma}\cos\theta_\nu + b_{\nu\sigma}\sin\theta_\nu, \quad \nu = 1, 2, \ldots, \mu$$

that is accurate to within the first nonlinear terms (only the resonance subsystem is written out).

The assumption that one of the resonances (2.8) is strong when $\nu = \varkappa$ i.e. that it results in the instability of the model system in the absence of other resonances, means that the system of equations

$$\begin{aligned} r_s^{\cdot} &= 2R_\varkappa Q^*_{\varkappa s}(\theta_\varkappa), \quad \sigma = 1, 2, \ldots, n_0 \\ r_{\varkappa\sigma}^{\cdot} &= 2R_\varkappa Q_{\varkappa\sigma}(\theta_\varkappa), \quad \sigma = 1, 2, \ldots, n_\varkappa \\ \theta_\varkappa^{\cdot} &= R_\varkappa\left(\sum_{\sigma=1}^{n_\varkappa}\frac{p_{\varkappa\sigma}}{r_{\varkappa\sigma}}Q'_{\varkappa\sigma} + \sum_{s=1}^{n_0}\frac{g_{\varkappa s}}{r_s}Q^{*\prime}_{\varkappa s}\right) \end{aligned} \tag{2.10}$$

has a particular unstable solution of the kind of increasing ray [3]. But then the model system (2.9) has a similar solution, since (2.10) is derived from it by setting $r_{\nu\sigma} = 0$ for all $\nu \neq \varkappa$.

This proves the following theorem.

Theorem 3. Weakness of each of the μ resonances (2.8) is the necessary condition of stability of the trivial solution of the model system (2.9).

Note. The correspondence of the unstable solution of the input model system to the indicated unstable particular solution of system (2.9) requires the exclusion from the analysis, as in the case of (2.6), the case $|P_\nu| \leqslant 1, \nu \neq \varkappa$, since it requires other methods of analysis.(*)

The stated necessary condition of stability of the model system in the case of resonance interaction of the kind (2.8) is, generally speaking, insufficient. It is possible to show that the interaction of several weak resonances linked by more than one common frequency can result in instability. An example of this is given in [2].

3. Example. As an example of the complex system in which can appear some of the investigated types of resonance interaction, we consider the translational-rotational motion of a geostationary satellite vehicle which can hover over some point of the Earth surface for a fairly extended time. For this it is sufficient to impart to it a constant in modulo reaction acceleration at a constant angle ψ to the axis of rotation of the Earth [7]. It was shown in [8] that on specific assumptions the equations of the translational-rotational motion of such satellites have a three-parameter set of solutions that correspond to the relative equilibrium of the vehicle in an orbital system of coordinates and to a uniform motion of its center of mass on a circular orbit.

The analysis of the stability of indicated particular solutions yields a system of 12-th order equations of perturbed motion, whose coefficients depend on parameters of the orbit and the geometry of the vehicle masses in a complex manner. Owing to this a complete analysis of stability can only be made on a computer. The variational equations are

$$x_s^{\cdot} = \sum_{i=1}^{12} a_{si}x_i, \quad s = 1, 2, \ldots, 12$$

*) The authors thank the reviewer for drawing their attention to this point.

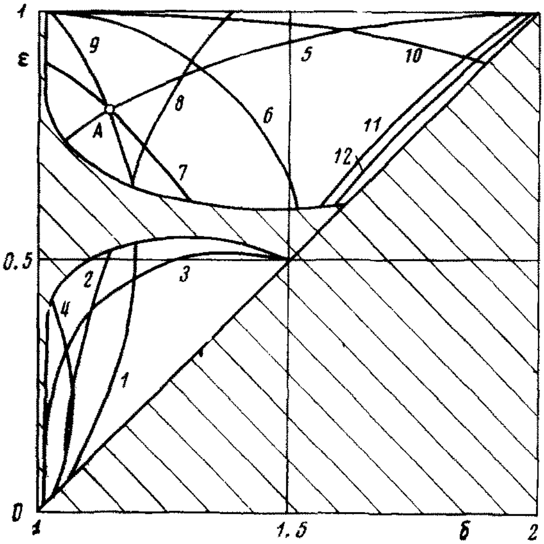

Fig. 1

where $x_1, \ldots, x_6$ are perturbations of radius ρ (for a stationary equatorial satellite $\rho = 1$), latitude φ and longitude of the vessel center of mass; x_7, x_8 and x_9 are perturbations of projections of the vehicle absolute angular velocity on the principal axis of inertia, and x_{10}, x_{11} , and x_{12} are perturbations of directional cosines of angles between the attached and an orbital system of coordinates. The non-zero elements of matrix $\| a_{si} \|$ are of the form

$$a_{14} = a_{25} = 1,\ a_{34} = -2,\ a_{35} = 2\operatorname{tg}\varphi,\ a_{39} = -\sin\varphi,$$
$$a_{3,10} = (1 - \rho\cos^2\varphi) / (2\rho\cos\varphi),\quad a_{41} = 2/\rho + \cos^2\varphi,\quad a_{42} = -\sin 2\varphi,$$
$$a_{43} = 2\cos^2\varphi,\ a_{4,11} = \sin 2\varphi / (2\cos\alpha),\ a_{51} = -\sin 2\varphi / 2,$$
$$a_{52} = -\cos 2\varphi,\ a_{53} = -\sin 2\varphi,\ a_{5,11} = (\rho\cos^2\varphi - 1) / (2\rho\cos\alpha),$$
$$a_{61} = 9(\varepsilon - \delta)\sin 2\alpha / (2\rho),\quad a_{67} = (\delta - \varepsilon)\sin(\alpha + \varphi),\quad a_{68} = (\delta - \varepsilon)\cdot$$
$$\cdot\cos(\alpha + \varphi),\ a_{6,11} = 3(\varepsilon - \delta)\cos 2\alpha / (\rho\cos\alpha),\ a_{76} = (\varepsilon - 1)\sin(\alpha + \varphi)/\delta,$$
$$a_{7,10} = 3(1 - \varepsilon)\cos\alpha / (p\delta),\ a_{86} = (1 - \delta)\cos(\alpha + \varphi) / \varepsilon,\ a_{8,10} =$$
$$3(1 - \delta)\sin\alpha / (\varepsilon\rho),\ a_{92} = -\cos\varphi,\ a_{93} = -\sin\varphi,\ a_{97} = -\sin\alpha,\ a_{98} =$$
$$\cos\alpha,\ a_{9,11} = \cos\varphi / \cos\alpha,\ a_{10,2} = -\sin\varphi,\ a_{10,3} = \cos\varphi,\ a_{10,7} =$$
$$-\cos\alpha,\ a_{10,8} = -\sin\alpha,\ a_{10,11} = \sin\varphi / \cos\alpha,\ a_{11,5} = \cos\alpha,\ a_{11,6} =$$
$$\cos\alpha,\ a_{11,9} = -\cos\varphi\cdot\cos\alpha,\ a_{11,10} = -\sin\varphi\cdot\cos\alpha$$
$$\varepsilon = C / A,\ \delta = B / A$$

where α is the angle of turn of the meridian plane containing the principal axes of inertia relative to the orbital system, and A, B and C are the squared radii of inertia of the vehicle relative to the principal axes which are considered to be constant.

The structure of matrix $\| a_{si} \|$ is such that the equation $\Delta(\lambda) \equiv \| a_{si} - \lambda\delta_{si} \| = 0$ has a pair of zero roots after the removal of which the characteristic polynomial $\Delta(\lambda)$ contains λ only of even powers and, consequently, stability is only possible in the critical case. (It can be shown (*) that by approximating the Earth potential by that of a triaxial ellipsoid the above pair of zero roots is converted to a pair of purely imaginary roots of small modulus.)

The results of calculations carried out on type M-220 computer for the determination the necessary conditions of stability in the plane of parameters ε , δ for $\rho = 1$ and $\varphi = 2^\circ$ (angle ψ was selected to satisfy the condition of minimum of re-

*) Myrzabekov, T., Stability of a stationary orbiting vehicle. Candidate's dissertation, Chimkent, 1975.

active acceleration) are shown in Fig. 1. In the unshaded region the characteristic equation has a pair of zero roots and five pairs of purely imaginary roots $\pm i\omega_s$ ($\omega_s > 0$, $s = 1,2, \ldots, 5$). In the shaded region there are roots with positive real parts, hence the motion is strictly Liapunov unstable.

When $\varphi = 0$ the frequencies ω_1, ω_2 and ω_3 determine only the rotational motion, while ω_4 and ω_5 define only the translational motion. These results have also shown that in the indicated stability region of linearized equations the following twelve third order resonance relationships (when $\varphi = 2°$ $\omega_4 = \omega_5$ with a considerable degree of accuracy):

1) $\omega_3 - 2\omega_2 = 0$, 2) $\omega_{4,5} - 2\omega_2 = 0$, 3) $\omega_1 - \omega_2 - \omega_3 = 0$, 4) $\omega_3 - \omega_2 - \omega_{4,5} = 0$, 5) $\omega_{4,5} - \omega_2 - \omega_3 = 0$, 6) $\omega_1 - \omega_2 - \omega_{4,5} = 0$, 7) $\omega_1 - 2\omega_3 = 0$, 8) $\omega_{4,5} - 2\omega_2 = 0$, 9) $\omega_3 - 2\omega_2 = 0$, 10) $\omega_2 - 2\omega_3 = 0$, 11) $\omega_1 - \omega_3 - \omega_5 = 0$, 12) $\omega_1 - \omega_3 - \omega_4 = 0$.

Resonance curves corresponding to these relationships along which instability of the system is possible are plotted in Fig. 1 (numbered in the same order). At intersection points (point A) of these curves the interaction of two or more resonances takes place. The problem of stability at these points is determined by the theorems proved above.

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Translated by J. J. D.